

The Higher Order Schwarzian Derivative: Its Applications for Chaotic Behavior and New Invariant Sufficient Condition of Chaos

G. HACIBEKİROĞLU, M. ÇAĞLAR and Y. POLATOĞLU

Abstract

The Schwarzian derivative of a function $f(x)$ which is defined in the interval (a, b) having higher order derivatives is given by

$$\mathcal{S}_{f(x)} = \left(\frac{f''(x)}{f'(x)} \right)' - \frac{1}{2} \left(\frac{f''(x)}{f'(x)} \right)^2.$$

A sufficient condition for a function to behave chaotically is that its Schwarzian derivative is negative. In this paper, we try to find a sufficient condition for a non-linear dynamical system to behave chaotically. The solution function of this system is a higher degree polynomial. We define n -th Schwarzian derivative to examine its general properties. Our analysis shows that the sufficient condition for chaotic behavior of higher order polynomial is provided if its highest order three terms satisfy an inequality which is invariant under the degree of the polynomial and the condition is represented by Hankel determinant of order 2. Also the n -th order polynomial can be considered to be the partial sum of real variable analytic function. Let this analytic function be the solution of non-linear differential equation, then the sufficient condition for the chaotical behavior of this function is the Hankel determinant of order 2 negative, where the elements of this determinant are the coefficient of the terms of $n, n-1, n-2$ in Taylor expansion.

1 Introduction

We need following definition,

Definition 1.1. If $f(x)$ is defined in the interval (a, b) and had higher order derivatives on this interval, then the Schwarzian derivative of f is defined by

$$\begin{aligned}\mathcal{S}_{f(x)} &= \left(\frac{f''(x)}{f'(x)} \right)' - \frac{1}{2} \left(\frac{f''(x)}{f'(x)} \right)^2 \\ &= \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2 \\ &= -2\sqrt{f'(x)} \frac{d^2}{dx^2} \left(\frac{1}{\sqrt{f'(x)}} \right).\end{aligned}$$

Note that the most important properties of Schwarzian derivative are the following [1], [2], [6]:

1. The Schwarzian derivative is invariant under the bilinear transformation of $f(x)$, i.e, if

$$T(f(x)) = \frac{af(x) + b}{cf(x) + d} \quad (1.1)$$

then

$$\mathcal{S}_{T(f(x))} = \mathcal{S}_{f(x)}$$

and there are no transformations possessing this property. We also have

$$\mathcal{S}_{g(t)} = \left(\frac{dx}{dt} \right)^2 \mathcal{S}_{f(x)} + \mathcal{S}_{\phi(t)},$$

where $x = \phi(t)$, $f(x) = f(\phi(t)) = g(t)$.

2. The equation $\mathcal{S}_{f(x)} = 0$ has the general solution $f(x) = \frac{ax + b}{cx + d}$ and the following hold:

$$\mathcal{S}_{f(x)} = -2\sqrt{f'(x)} \frac{d^2}{dx^2} \left(\frac{1}{\sqrt{f'(x)}} \right) = 0 \Rightarrow \frac{d^2}{dx^2} \left(\frac{1}{\sqrt{f'(x)}} \right) = 0 \Rightarrow$$

$$f'(x) = \frac{1}{(ax + b)^2} \Rightarrow f(x) = \frac{c}{(ax + b)^2}.$$

Now we define the differential operator [3], [4], [5], [7]

$$\begin{aligned}
\mathcal{S}_{f_n(x)} &= \left(\frac{f^{(n+1)}(x)}{f^{(n)}(x)} \right)' - \frac{1}{(n+1)} \left(\frac{f^{(n+1)}(x)}{f^{(n)}(x)} \right)^2 \\
&= \left(\frac{f^{(n+2)}(x)}{f^{(n)}(x)} \right) - \frac{(n+2)}{(n+1)} \left(\frac{f^{(n+1)}(x)}{f^{(n)}(x)} \right)^2 \\
&= -(n+1) {}^{n+1}\sqrt{f^{(n)}(x)} \frac{d^2}{dx^2} \left(\frac{1}{\sqrt{f^{(n)}(x)}} \right)
\end{aligned} \tag{1.2}$$

which is called the n -th Schwarzian derivative.

The transformation $T_{f(x)}$ of $f(x)$ which leaves its n -th Schwarzian derivative invariant can be obtained in the following way: If $T_{f(x)} = f_1(x)$ then the relation

$$\begin{aligned}
-(n+1) {}^{n+1}\sqrt{f^{(n)}(x)} \frac{d^2}{dx^2} \left(\frac{1}{{}^{n+1}\sqrt{f^{(n)}(x)}} \right) &= -(n+1) {}^{n+1}\sqrt{f_1^{(n)}(x)} \frac{d^2}{dx^2} \left(\frac{1}{{}^{n+1}\sqrt{f_1^{(n)}(x)}} \right) \Rightarrow \\
{}^{n+1}\sqrt{f^{(n)}(x)} \frac{d^2}{dx^2} \left(\frac{1}{{}^{n+1}\sqrt{f^{(n)}(x)}} \right) &= {}^{n+1}\sqrt{f_1^{(n)}(x)} \frac{d^2}{dx^2} \left(\frac{1}{{}^{n+1}\sqrt{f_1^{(n)}(x)}} \right),
\end{aligned}$$

having defined

$$u(x) = \frac{1}{{}^{n+1}\sqrt{f^{(n)}(x)}}, \quad v(x) = \frac{1}{{}^{n+1}\sqrt{f_1^{(n)}(x)}} \tag{1.3}$$

yields

$$\frac{u''(x)}{u(x)} = \frac{v''(x)}{v(x)} \quad \text{or} \quad v''(x) - \frac{u''(x)}{u(x)}v(x) = 0. \tag{1.4}$$

So, the general solution of (1.4) is given by

$$v(x) = u(x) \left(a_1 + a_2 \int_0^x \frac{dt}{(u(t))^2} \right), \tag{1.5}$$

where a_1 and a_2 are constants. On the other hand from (1.3) and (1.5) we have

$$f_1^{(n)}(x) = \frac{f^{(n)}(x)}{\left(a_1 + a_2 \int_0^x (f^{(n)}(x))^{\frac{2}{n+1}} dx \right)^{n+1}},$$

and hence

$$f_1(x) = T(f(x)) = \underbrace{\int \int \int \cdots \int}_{n\text{-times}} \frac{f^{(n)}(x)}{\left(a_1 + a_2 \int_0^x (f^{(n)}(x))^{\frac{2}{n+1}} dx\right)^{n+1}} \underbrace{dx dx dx \cdots dx}_{n\text{-times}}. \quad (1.6)$$

For $n = 1$, (1.6) becomes the bilinear transformation of equation (1.1).

The Hankel matrix H of the integer sequence $A = \{a_1, a_2, a_3, \dots\}$ is the infinite matrix [8], [9]

$$H = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & a_6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore the Hankel matrix H_n of order of A is the upper left $(n \times n)$ submatrix of H , and h_n , the Hankel determinant of order n of A is the determinant of the corresponding Hankel matrix of order n , $h_n = \det(H_n)$. For example the Hankel matrix of order 4 of the Fibonacci sequence $1, 1, 2, 3, 5, \dots$, is

$$H_4 = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 13 \end{pmatrix},$$

with 4-th order Hankel determinant $h_4 = 0$.

2 Main Results

In this section we will give a sufficient condition for the chaotic behavior of an n -th degree polynomial. Let be a polynomial of degree n with real coefficient where $a_1 \neq 0$ having roots x_i , $i = 1, 2, 3, \dots, n$

$$P(x) = a_1 x^n + a_2 x^{n-1} + a_3 x^{n-2} + \cdots + a_{n-1} x + a_n. \quad (2.1)$$

Theorem 2.1. *The Schwarzian derivative of the polynomial of (2.1) is*

$$\begin{aligned} \mathcal{S}_{P(x)} = & \frac{(n+2)!(n+1)a_1 \left(\frac{(n+2)!}{2!}a_1x^2 + (n+1)!a_2x + n!a_3 \right)}{(n+1) \left(\frac{(n+2)!}{2!}a_1x^2 + (n+1)!a_2x + n!a_3 \right)^2} \\ & - \frac{(n+2) \left((n+2)!a_1x + (n+1)!a_2 \right)^2}{(n+1) \left(\frac{(n+2)!}{2!}a_1x^2 + (n+1)!a_2x + n!a_3 \right)^2}. \end{aligned} \quad (2.2)$$

Proof. The proof (2.2) we will use the induction principle. Using formula (1.2), we get for $n = 1$, $P(x) = a_1x^3 + a_2x^2 + a_3x + a_4$, i.e.,

$$\begin{aligned} \mathcal{S}_{P(x)} &= \left(\frac{P^{(n+1)}(x)}{P^{(n)}(x)} \right)' - \frac{1}{2} \left(\frac{P^{(n+1)}(x)}{P^{(n)}(x)} \right)^2 \Rightarrow \\ \mathcal{S}_{P(x)} &= \frac{3!2a_1 \left(\frac{3!}{2!}a_1x^2 + 2!a_2x + 1!a_3 \right) - 3 \left(3!a_1x + 2!a_2x \right)^2}{2 \left(\frac{3!}{2!}a_1x^2 + 2!a_2x + 1!a_3 \right)^2}, \end{aligned}$$

so, (2.2) is true for $n = 1$.

Suppose that this result is true for $n = k$. Then we have, $P(x) = a_1x^{k+2} + a_2x^{k+1} + a_3x^k + a_4x^{k-1} + \dots + a_{k+1}$,

$$\begin{aligned} \mathcal{S}_{P(x)} &= \left(\frac{P^{(k+1)}(x)}{P^{(k)}(x)} \right)' - \frac{1}{k+1} \left(\frac{P^{(k+1)}(x)}{P^{(k)}(x)} \right)^2 \Rightarrow \\ \mathcal{S}_{P(x)} &= \frac{(k+2)!(k+1)a_1 \left(\frac{(k+2)!}{2!}a_1x^2 + (k+1)!a_2x + k!a_3 \right)}{(k+1) \left(\frac{(k+2)!}{2!}a_1x^2 + (k+1)!a_2x + k!a_3 \right)^2} \\ & - \frac{(k+2) \left((k+2)!a_1x + (k+1)!a_2 \right)^2}{(k+1) \left(\frac{(k+2)!}{2!}a_1x^2 + (k+1)!a_2x + k!a_3 \right)^2}. \end{aligned}$$

From the induction hypothesis, we then have, $P(x) = a_1x^{k+3} + a_2x^{k+2} + a_3x^{k+1} + \dots + a_{k+1}$,

$$\mathcal{S}_{P(x)} = \left(\frac{P^{(k+2)}(x)}{P^{(k+1)}(x)} \right)' - \frac{1}{k+2} \left(\frac{P^{(k+2)}(x)}{P^{(k+1)}(x)} \right)^2 \Rightarrow$$

$$\mathcal{S}_{P(x)} = \frac{(k+3)!(k+2)a_1 \left(\frac{(k+3)!}{2!}a_1x^2 + (k+2)!a_2x + (k+1)!a_3 \right)}{(k+2) \left(\frac{(k+3)!}{2!}a_1x^2 + (k+2)!a_2x + (k+1)!a_3 \right)^2} - \frac{(k+3) \left((k+3)!a_1x + (k+2)!a_2 \right)^2}{(k+2) \left(\frac{(k+3)!}{2!}a_1x^2 + (k+2)!a_2x + (k+1)!a_3 \right)^2}.$$

which is the desired conclusion. \square

Theorem 2.2. *A necessary condition for an n -th degree polynomial to be chaotic is that*

$$a_2^2 - \frac{2n+4}{2n+3}a_1a_3 > 0 \quad \text{for all } n = 1, 2, 3, \dots$$

Proof. Using the result of the Theorem 2.1, we get

$$\begin{aligned} \mathcal{S}_{P(x)} < 0 \Rightarrow & \left[\frac{(n+2)!(n+1)}{2!} - (n+2)!(n+2) \right] a_1^2x^2 \\ & + [(n+1)!(n+2)!(n+1) - 2(n+1)!(n+3)!(n+2)] a_1a_2x \\ & + [(n)!(n+2)!(n+1)a_1a_3 - ((n+1)!)^2(n+2)a_2^2] < 0 \end{aligned}$$

or

$$\begin{aligned} & \left[((n+2)!)^2(n+2) - \frac{((n+2)!)^2(n+1)}{2!} \right] a_1^2x^2 \\ & + [2(n+1)!(n+2)!(n+2) - (n+1)!(n+2)!(n+1)] a_1a_2x^2 \\ & + [((n+1)!)^2n!(n+2)a_2^2 - (n)!(n+2)!(n+1)a_1a_3] > 0. \end{aligned} \tag{2.3}$$

If we take

$$\begin{aligned} A &= \left[((n+2)!)^2(n+2) - \frac{((n+2)!)^2(n+1)}{2!} \right] a_1^2, \\ B &= [2(n+1)!(n+2)!(n+2) - (n+1)!(n+2)!(n+1)] a_1a_2, \\ C &= [((n+1)!)^2n!(n+2)a_2^2 - (n)!(n+2)!(n+1)a_1a_3], \end{aligned}$$

then the inequality (2.3) can be written in the form

$$Ax^2 + Bx + C > 0. \tag{2.4}$$

In order condition (2.4) to be satisfied, the discriminant of the polynomial should be negative and the coefficient of x^2 should be positive, i.e,

$$A = \left[((n+2)!)^2(n+2) - \frac{((n+2)!)^2(n+1)}{2!} \right] a_1^2 > 0 \quad (2.5)$$

i.e.,

$$B^2-4AC = [(n+2)!(n+3) - 2(n+2)!(n+3)a_2 - 2(n+2)!(n+3)a_1a_3] < 0.$$

If we write (2.5) for $n = 1, 2, 3, \dots$ we obtain following inequalities

$$\begin{aligned} n=1, & \quad a_2^2 - 3a_1a_3 > 0 \Rightarrow a_2^2 - 3a_1a_3 > 0, \\ n=2, & \quad 7a_2^2 - 8a_1a_3 > 0 \Rightarrow a_2^2 - \frac{8}{7}a_1a_3 > 0, \\ n=3, & \quad 9a_2^2 - 10a_1a_3 > 0 \Rightarrow a_2^2 - \frac{10}{9}a_1a_3 > 0, \\ n=4, & \quad 11a_2^2 - 12a_1a_3 > 0 \Rightarrow a_2^2 - \frac{12}{11}a_1a_3 > 0, \\ n=5, & \quad 13a_2^2 - 14a_1a_3 > 0 \Rightarrow a_2^2 - \frac{14}{13}a_1a_3 > 0, \\ & \vdots & \qquad \qquad \qquad \vdots & \qquad \qquad \qquad \vdots \end{aligned}$$

so, by induction, we have

$$a_2^2 - \frac{2n+4}{2n+3} > 0, \quad n \geq 2.$$

□

As a result, a necessary condition for the chaotic behavior of the polynomial $P(x)$, is that

$$a_2^2 - \frac{2n+4}{2n+3}a_1a_3 > 0, \quad n \geq 2.$$

Furthermore, letting n go to infinity, we have $a_2^2 - a_1 a_3 > 0$. This implies that a real analytic function whose Maclaurin series expansion is $\sum_{n=0}^{\infty} a_n x^n$ behaves chaotically whenever $a_2^2 - a_1 a_3 > 0$. If we use the Hankel determinant, this condition can be written in the form

$$\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} < 0.$$

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GÜRSEL. HACİBEKİROĞLU

Department of Physics,
TC İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: g.hacibekiroglu@iku.edu.tr

MERT ÇAĞLAR

Department of Mathematics and Computer Science,
TC İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: m.caglar@iku.edu.tr

YAŞAR POLATOĞLU

Department of Mathematics and Computer Science,
TC İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: y.polatoglu@iku.edu.tr